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EQUATIONS OF AN ELASTIC ANISOTROPIC LAYER

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Differential equations of an elastic orthotropic layer are constructed on the basis of expansion of the solutions of the elasticity theory in terms of the Legendre polynomials. The order of the system of differential equations is independent of the form of the boundary conditions on the layer surfaces, which allows a correct formulation of conditions on contact surfaces.

Key words: *orthotropic elastic layer, Legendre polynomials.*

Introduction. In reducing the three-dimensional problem of the elasticity theory to a two-dimensional problem (theory of shells), either hypotheses of a kinematic and force character [1] or expansions in a certain full system of functions [2] are used. Equations of the theory of shells with the use of the Kirchhoff–Love hypotheses are normally constructed for the case where the forces on the shell surfaces are specified. This complicates the solution of contact problems on the basis of such equations and often leads to nonphysical effects. Differential equations of an elastic layer, whose order is independent of the form of the boundary conditions on the layer surfaces, which ensures well-posedness of the contact problems, are constructed in [3, 4] on the basis of expansions in terms of the Legendre polynomials. The equations of the layer in the first approximation are reduced to a system of ordinary differential equations with constant coefficients. Generic solutions of these equations for an isotropic elastic layer of constant thickness are given in [4], and solutions of some contact problems are described in [4–6]. The problem of bending of a three-layer orthotropic beam was solved in [7] on the basis of the elastic layer equations in the first approximation. A comparison of solutions of contact problems on the basis of the elastic layer equations in the first approximation and solutions obtained by the elasticity theory equations revealed good agreement of results obtained by approximate equations and elasticity theory equations [5]. The approach to constructing approximate equations proposed in [3] was used in [8] to construct equations of an elastic layer of variable thickness. The elastic layer equations in the first approximation can be used for the numerical solution of two-dimensional problems of the elasticity theory. A numerical algorithm for solving two-dimensional problems of the elasticity theory by the method of layers was suggested in [9].

Differential equations of an anisotropic elastic layer in the first approximation are given in the present paper.

1. Equations of the Two-Dimensional Problem of the Elasticity Theory. We write the equations of the two-dimensional problem of the elasticity theory in a rectangular domain Ω: $\{-l \leqslant x_1 \leqslant l,\ -h/2 \leqslant x_2 \leqslant h/2\}$:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad \sigma_{ij} = a_{ijmn}\varepsilon_{mn}, \qquad \varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{1.1}$$

Here h is the layer thickness, $2l$ is the layer length, σ_{ij} and ε_{ij} are the stresses and strains, respectively, u_i are the displacements, and f_i and a_{ijmn} are specified piecewise-continuous functions x_1 and x_2; the coefficients a_{ijmn} satisfy the conditions

$$a_{ijmn}\varepsilon_{ij}\varepsilon_{mn} - c\varepsilon_{ij}\varepsilon_{ij} \geqslant 0, \qquad a_{ijmn} = a_{jimn} = a_{ijnm},$$

where c is a nonnegative constant; the subscripts i and j take the values 1 and 2; summation is performed over the dummy subscripts. At the boundary of the domain Ω, we impose the boundary conditions of the form

$$c_{i1}^{\pm} u_i + d_{i1}^{\pm}\sigma_{i1} = \varphi_{i1}^{\pm} \qquad \text{for} \quad x_1 = \pm l; \tag{1.2}$$

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$$c_{i2}^{\pm}u_i + d_{i2}^{\pm}\sigma_{i2} = \varphi_{i2}^{\pm} \qquad \text{for} \quad x_2 = \pm h/2, \tag{1.3}$$

where $c_{i2}^{\pm}(x_1)$, $d_{i2}^{\pm}(x_1)$, $\varphi_{i2}^{\pm}(x_1)$, and $\varphi_{i1}^{\pm}(x_2)$ are specified piecewise-continuous functions; $c_{i1}^{\pm}$ and $d_{i1}^{\pm}$ are specified constants that satisfy the conditions

$$|c_{ij}^{\pm}| + |d_{ij}^{\pm}| \neq 0, \qquad c_{ij}^{+}d_{ij}^{+} \geqslant 0, \qquad c_{ij}^{-}d_{ij}^{-} \leqslant 0. \tag{1.4}$$

Inequalities (1.4) ensure dissipation of the boundary conditions (1.2) and (1.3).

2. Expansion of Stresses and Displacements in Series in Terms of the Legendre Polynomials. Basic Boundary Conditions. Let the stresses and displacements be expanded in series in terms of the Legendre polynomials

$$\sigma_{ij} = \sum_{k=0}^{\infty} \sigma_{ij}^{k} P_k(\zeta), \qquad u_i = \sum_{k=0}^{\infty} u_i^k P_k(\zeta), \tag{2.1}$$

where $\zeta = 2x_2/h$, $P_k(\zeta)$ are the Legendre polynomials, and

$$\sigma_{ij}^{k} = \frac{1+2k}{2}\int_{-1}^{+1} \sigma_{ij} P_k(\zeta)\, d\zeta; \qquad u_i^k = \frac{1+2k}{2}\int_{-1}^{+1} u_i P_k(\zeta)\, d\zeta.$$

It follows from (2.1) that

$$\sigma_{11}^{0} = T_{11}/h, \qquad \sigma_{11}^{1} = 6M_{11}/h^2, \qquad \sigma_{12}^{0} = T_{21}/h, \tag{2.2}$$

where

$$T_{11} = \int_{-h/2}^{h/2} \sigma_{11}\, dx_2, \qquad M_{11} = \int_{-h/2}^{h/2} \sigma_{11} x_2\, dx_2, \qquad T_{21} = \int_{-h/2}^{h/2} \sigma_{21}\, dx_2 \tag{2.3}$$

are the force, moment, and lateral shear force in the layer cross section $x_1 = \text{const}$, respectively. In (2.1), the first two terms of the series for u_1 and the first term of the series for u_2 correspond to the displacement of the layer as a stiff whole. If the stresses and displacements are presented in the form of series (2.1), the boundary conditions (1.2) can be written as conditions on the coefficients of these series $u_i^k(x_1)$ and $\sigma_{ij}^k(x_1)$:

$$c_{i1}^{\pm}u_i^k + d_{i1}^{\pm}\sigma_{i1}^{k} = (\varphi_{i1}^{\pm})^k, \qquad k = 0, 1, 2, \ldots . \tag{2.4}$$

In (2.4), $(\varphi_{i1}^{\pm})^k$ are the coefficients of the Legendre series of the functions $\varphi_{i1}^{\pm}$.

If the layer thickness is small ($h \ll l$), then, by virtue of the Saint-Venant principle, conditions (2.4) can be divided into two groups: 1) conditions affecting the solution for all $|x_1| \leqslant l$; 2) conditions affecting the solution only in the neighborhood of the cross sections $x_1 = \pm l$. *The conditions affecting the solution for all $|x_1| \leqslant l$ can be called the basic conditions.* According to the Saint-Venant principle, such conditions contain quantities (2.2), i.e.,

$$\begin{aligned} &c_{11}^{\pm}u_1^k + d_{11}^{\pm}\sigma_{11}^{k} = (\varphi_{11}^{\pm})^k, \qquad k = 0, 1, \\ &c_{21}^{\pm}u_2^0 + d_{21}^{\pm}\sigma_{21}^{0} = (\varphi_{21}^{\pm})^0 \qquad \text{for} \quad x_1 = \pm l. \end{aligned} \tag{2.5}$$

3. Equations of the One-Dimensional Problem of the First Approximation. In constructing equations in the first approximation, we require that the solution of the one-dimensional problem is possible under arbitrary conditions (2.5) [arbitrary values of $(\varphi_{11}^{\pm})^k$ ($k = 0, 1$) and $(\varphi_{21}^{\pm})^0$ and arbitrary values of $c_{i1}^{\pm}$ and $d_{i1}^{\pm}$ admitted by inequalities (1.4)]. Since there are six conditions (2.5), the minimum order of the system of the one-dimensional problem of the first approximation should be six, and the system should not contain any other derivatives except for

$$\frac{du_1^k}{dx_1}, \quad \frac{d\sigma_{11}^k}{dx_1}, \quad k = 0, 1, \qquad \frac{du_2^0}{dx_1}, \quad \frac{d\sigma_{21}^0}{dx_1}. \tag{3.1}$$

Thus, in the series for the derivatives

$$\frac{\partial\sigma_{11}}{\partial x_1}, \quad \frac{\partial\sigma_{21}}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_1}, \tag{3.2}$$

entering into the equilibrium equations [the first group of Eqs. (1.1)], we leave only terms that contain derivatives (3.1). Thus, derivatives (3.2) are replaced by the derivatives

$$\frac{\partial \sigma'_{11}}{\partial x_1}, \quad \frac{\partial \sigma'_{21}}{\partial x_1}, \quad \frac{\partial u'_1}{\partial x_1}, \quad \frac{\partial u'_2}{\partial x_1},$$

where

$$\sigma'_{11} = \sum_{k=0}^{1} \sigma_{11}^k P_k(\zeta), \quad \sigma'_{21} = \sigma_{21}^0, \quad u'_1 = \sum_{k=0}^{1} u_1^k P_k(\zeta), \quad u'_2 = u_2^0. \tag{3.3}$$

In the series for the derivatives $\partial\sigma_{12}/\partial x_2$ and $\partial\sigma_{22}/\partial x_2$ in Eqs. (1.1), we leave only those terms that provide the correspondence

$$\frac{\partial \sigma_{12}}{\partial x_2} \sim \frac{\partial \sigma'_{11}}{\partial x_1}, \qquad \frac{\partial \sigma_{22}}{\partial x_2} \sim \frac{\partial \sigma'_{21}}{\partial x_1}. \tag{3.4}$$

In (3.4) and below, the tilde indicates an identical degree of the polynomials with respect to ζ. Therefore, in the one-dimensional problem of the first approximation, σ_{12} and σ_{22} in the equilibrium equations from (1.1) are replaced by polynomial segments

$$\sigma'_{12} = \sum_{k=0}^{2} \sigma_{12}^k P_k(\zeta), \qquad \sigma'_{22} = \sum_{k=0}^{1} \sigma_{22}^k P_k(\zeta). \tag{3.5}$$

The mass forces f_1 and f_2 are replaced by the segments f'_1 and f'_2 of the Legendre polynomial series so that the following correspondence holds:

$$f'_1 \sim \frac{\partial \sigma'_{11}}{\partial x_1}, \qquad f'_2 \sim \frac{\partial \sigma'_{21}}{\partial x_1}.$$

Thus, we have

$$f'_1 = \sum_{k=0}^{1} f_1^k P_k(\zeta), \qquad f'_2 = f_2^0, \qquad f_i^k = \frac{1+2k}{2} \int_{-1}^{+1} f_i P_k(\zeta)\, d\zeta, \quad k = 1, 2. \tag{3.6}$$

In accordance with approximations (3.3), (3.5), and (3.6), the equilibrium equations from (1.1) are replaced by the equations

$$\frac{\partial \sigma'_{ij}}{\partial x_j} + f'_i = 0. \tag{3.7}$$

In (3.3), (3.5), the coefficients σ_{ij}^k are the sought functions of the variable x_1.

Designating

$$\sigma'_{12}(\zeta = \pm 1) = \sigma_{12}^{\pm}, \qquad \sigma'_{22}(\zeta = \pm 1) = \sigma_{22}^{\pm},$$

we obtain from (3.5)

$$\sigma_{12}^2 = (\sigma_{12}^+ + \sigma_{12}^-)/2 - \sigma_{12}^0, \qquad \sigma_{12}^1 = (\sigma_{12}^+ - \sigma_{12}^-)/2,$$

$$\sigma_{22}^1 = (\sigma_{22}^+ - \sigma_{22}^-)/2, \qquad \sigma_{22}^0 = (\sigma_{22}^+ - \sigma_{22}^-)/2.$$

Using these relations and (2.3), we can write Eqs. (3.7) as

$$\frac{dT_{11}}{dx_1} + \sigma_{12}^+ - \sigma_{12}^- + f_1^0 h = 0, \qquad \frac{dM_{11}}{dx_1} + \frac{\sigma_{12}^+ + \sigma_{12}^-}{2}\, h - \sigma_{12}^0 h + \frac{1}{6}\, h^2 f_1^1 = 0,$$

$$\frac{dT_{21}}{dx_1} + \sigma_{22}^+ - \sigma_{22}^- + f_2^0 h = 0. \tag{3.8}$$

If we assume that $\sigma_{21}^0 = \sigma_{12}^0$ and, hence, $\sigma_{12}^0 = Q$, Eqs. (3.8) are equations of equilibrium of an element loaded by the mass forces f_i and surface forces $\sigma_{12}^{\pm}$ and $\sigma_{22}^{\pm}$; the size of the element is infinitesimal in the x_1 direction and equals h in the x_2 direction. Thus, in the one-dimensional problem of the first approximation, the requirement that the stresses should satisfy the equations of equilibrium of arbitrary infinitesimal elements is replaced by a less rigorous requirement that the stresses should satisfy the conditions of equilibrium of elements whose size is infinitesimal only in the x_1 direction and finite in the x_2 direction.

In addition to approximations of displacements u_1' and u_2' [relations (3.3)], the following approximations are used in the equations of the one-dimensional problem:

$$u_1'' = \sum_{k=0}^{3} u_1^k P_k(\zeta), \qquad u_2'' = \sum_{k=0}^{2} u_2^k P_k(\zeta). \tag{3.9}$$

The length of polynomial segments in (3.9) is determined by the relations

$$\frac{\partial u_1''}{\partial x_2} \sim \sigma_{12}', \qquad \frac{\partial u_2''}{\partial x_2} \sim \sigma_{22}'.$$

Approximations (3.9) are used in substituting the derivatives $\partial u_1/\partial x_2$ and $\partial u_2/\partial x_2$ in the last group of Eqs. (1.1) by the derivatives $\partial u_1''/\partial x_2$ and $\partial u_2''/\partial x_2$.

The strains in the one-dimensional problem of the first approximation are expressed via the segments u_i' and u_i'' as follows:

$$\varepsilon_{11} = \frac{\partial u_1'}{\partial x_1}, \qquad 2\varepsilon_{12} = \frac{\partial u_1''}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}, \qquad \varepsilon_{22} = \frac{\partial u_2''}{\partial x_2}. \tag{3.10}$$

The stresses inside the layer are calculated by the formulas

$$\sigma_{ij} = a_{ijmn}\varepsilon_{mn}, \tag{3.11}$$

where ε_{mn} is determined by relations (3.10). Though the strains are polynomials in terms of the x_2 coordinate, the stresses can be other than polynomials if the elasticity coefficients depend on the coordinates x_1 and x_2, e.g., in the case of an inhomogeneous material.

It follows from (3.3), (3.5), (3.10), and (3.11) that the coefficients σ_{ij}^k and u_i^k are related as

$$\sigma_{ij}^k = \frac{1+2k}{2} \int_{-1}^{+1} a_{ijmn}\varepsilon_{mn} P_k(\zeta)\, d\zeta. \tag{3.12}$$

The boundary conditions (1.3) in the one-dimensional problem are replaced by the conditions

$$c_{i2}^{\pm} u_i'' + d_{i2}^{\pm}\sigma_{i2}' = \varphi_{i2}^{\pm} \quad (i = 1, 2) \quad \text{for} \quad x_2 = \pm h/2. \tag{3.13}$$

Equations (3.3), (3.5), (3.7), and (3.9)–(3.13) and the boundary conditions (2.5) form a closed system of one-dimensional equations of the first approximation for the elastic layer. Note, the equilibrium equations (3.8) can be written as

$$\int_{-1}^{+1} \Big[\frac{\partial \sigma_{1i}}{\partial x_i} + f_1\Big] P_k(\zeta)\, d\zeta = 0, \qquad \int_{-1}^{+1} \Big[\frac{\partial \sigma_{2i}}{\partial x_i} + f_2\Big] d\zeta = 0 \qquad (k = 0, 1),$$

and Eqs. (3.11) can be written as

$$\int_{-1}^{+1} [\sigma_{11}' - a_{11ij}\varepsilon_{ij}] P_k(\zeta)\, d\zeta = 0, \qquad \int_{-1}^{+1} [\sigma_{22}' - a_{22ij}\varepsilon_{ij}] P_k(\zeta)\, d\zeta = 0,$$

$$\int_{-1}^{+1} [\sigma_{12}' - a_{12ij}\varepsilon_{ij}] P_k(\zeta) d\zeta = 0, \qquad \sigma_{21}^0 = \sigma_{12}^0.$$

The solution of the one-dimensional problem reduces to the solution of a system of differential equations for the functions

$$u_1^0,\ u_1^1,\ u_2^0,\ \sigma_{11}^0,\ \sigma_{11}^1,\ \sigma_{21}^0. \tag{3.14}$$

This system has the fourth order regardless of the form of the boundary conditions on the layer surfaces $x_2 = \pm h/2$.

The functions (3.14) are called the basic functions, and the functions σ_{22}^0, σ_{22}^1, σ_{12}^1, σ_{12}^2, u_1^2, u_1^3, u_2^1, and u_2^2 are called the additional functions.

We can show that the solution of the one-dimensional problem satisfies the energy identity

$$\int_\Omega \sigma_{ij}\varepsilon_{ij}\,d\Omega = \int_\Omega \Big(f_i' u_i' + \frac{\partial(\sigma_{i1}' u_i')}{\partial x_1} + \frac{\partial(\sigma_{i2}' u_i'')}{\partial x_2}\Big)\,d\Omega. \tag{3.15}$$

Identity (3.15) allows us to prove the uniqueness of a certain class of contact problems for the elastic layer [4].

4. Transition to Dimensionless Variables in the Equations of the One-Dimensional Problem. In what follows, we use the dimensionless quantities

$$\bar\sigma_{ij} = \frac{\sigma_{ij}}{\sigma_0}, \qquad \bar\varepsilon_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}, \qquad \varepsilon_0 = \frac{\sigma_0}{\mu}, \qquad \bar u_i = \frac{2u_i}{h\varepsilon_0},$$
$$\xi = \frac{x_1}{L_0}, \qquad \zeta = \frac{2x_2}{h}, \qquad \eta = \frac{h}{2L_0}, \qquad \bar f_i^0 = \frac{f_i^0 h}{2\sigma_0}. \tag{4.1}$$

The segments of the Legendre polynomial series (3.3), (3.5), and (3.9) for dimensionless stresses and displacements are written in the form

$$\bar\sigma_{11}' = t_{11} + m_{11}P_1, \qquad \bar\sigma_{12}' = t_{12} + m_{12}P_1 + r_{12}P_2,$$
$$\bar\sigma_{21}' = t_{21}, \qquad \bar\sigma_{22}' = t_{22} + m_{22}P_1; \tag{4.2}$$

$$\bar u_1' = u_0 + u_1 P_1, \qquad \bar u_2' = v_0,$$
$$\bar u_1'' = u_0 + u_1P_1 + u_2P_2 + u_3P_3, \qquad \bar u_2'' = v_0 + v_1P_1 + v_2P_2. \tag{4.3}$$

In (4.2) and (4.3),

$$t_{11} = \frac{T_{11}}{h\sigma_0}, \qquad t_{21} = t_{12} = \frac{T_{21}}{h\sigma_0}, \qquad m_{11} = \frac{6M_{11}}{h^2\sigma_0},$$
$$u_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h}\,dx_2, \qquad u_1 = \frac{6}{h^2\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_1}{h}x_2\,dx_2, \qquad v_0 = \frac{1}{h\varepsilon_0}\int_{-h/2}^{h/2} \frac{u_2}{h}\,dx_2. \tag{4.4}$$

We write the relation between the dimensionless stresses and strains:

$$\bar\sigma_1 = \alpha_1(\bar\varepsilon_1 + \gamma_1\bar\varepsilon_2), \qquad \bar\sigma_2 = \alpha_2(\bar\varepsilon_2 + \gamma_2\bar\varepsilon_1), \qquad \bar\sigma_{12} = 2\bar m\varepsilon_{12}. \tag{4.5}$$

For a transversely isotropic material, the elastic constants in relations (4.5) have the form

$$\alpha_1 = \frac{E_2}{\sigma_0}\frac{n(1-n\nu_2^2)}{(1+\nu_1)(1-\nu_1-2n\nu_2^2)}, \qquad \alpha_2 = \alpha_1\frac{1-\nu_1^2}{n(1-n\nu_2^2)},$$
$$\gamma_1 = \frac{\nu_2(1+\nu_1)}{1-n\nu_2^2}, \qquad \gamma_2 = \frac{n\nu_2}{1-\nu_1}, \qquad \bar m = m\frac{E_2}{\sigma_0} \tag{4.6}$$

in the case of plane deformation or

$$\alpha_1 = \frac{E_2 n}{\sigma_0(1-n\nu_2)}, \quad \alpha_2 = \frac{\alpha_1}{n}, \quad \gamma_1 = \nu_2, \quad \gamma_2 = n\nu_2, \quad \bar m = m\frac{E_2}{\sigma_0} \tag{4.7}$$

for the plane-stressed state. The elastic constants E_1 and ν_1 in (4.5) characterize the material behavior in the anisotropy plane, and the elastic constants E_2, G_2, and ν_2 characterize its behavior in the direction orthogonal to the isotropy plane, $n = E_1/E_2$, and $m = G_2/E_2$.

For an orthotropic material, the elastic constants in Eqs. (4.5) for a plane-stressed state are written as

$$\alpha_1 = \frac{E_x}{\sigma_0(1-\nu_{xy}\nu_{xy})}, \qquad \alpha_2 = \frac{E_y}{\sigma_0(1-\nu_{xy}\nu_{xy})},$$
$$\gamma_1 = \nu_{xy}, \qquad \gamma_2 = \nu_{yx}, \qquad \bar m = m\frac{G_{xy}}{\sigma_0}, \qquad \frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}.$$

For an isotropic material, Eqs. (4.6), (4.7) for $E_1 = E_2$, $G_1 = G_2$, and $\nu_1 = \nu_2$ yield the relations

$$\alpha_1 = \alpha_2 = \alpha = 2/(1-\gamma), \tag{4.8}$$

where $\gamma = \nu/(1-\nu)$ in the case of plane deformation and $\gamma = \nu$ for the plane-stressed state.

The system of differential equations for the coefficients of expansions (4.2) and (4.3) in dimensionless variables is written in the following form:

$$\begin{gathered}
\eta t'_{11} + (\bar\sigma^+_{12} - \bar\sigma^-_{12})/2 + \bar f^0_1 = 0, \qquad \eta t'_{12} + (\bar\sigma^+_2 - \bar\sigma^-_2)/2 + \bar f^0_2 = 0,\\
\eta m'_{11} - 3t_{12} + 3(\bar\sigma^+_{12} + \bar\sigma^-_{12})/2 + \bar f^1_1 = 0,\\
t_{11} = \alpha_1(\eta u'_0 + \gamma_1 v_1), \qquad t_{22} = \alpha_2(\gamma_2 \eta u'_0 + v_1),\\
m_{11} = \alpha_1(\eta u'_1 + 3\gamma_1 v_2), \qquad m_{22} = \alpha_2(\gamma_2\eta u'_1 + 3v_2),\\
t_{12} = m(\eta v'_0 + u_1 + u_3), \qquad m_{12} = 3mu_2, \qquad r_{12} = 5mu_3.
\end{gathered} \tag{4.9}$$

System (4.9) of 10 equations with respect to 14 coefficients of expansions (4.2) and (4.3) is closed by four conditions on the layer surfaces for $\zeta = \pm 1$:

$$c^\pm_{i2} u''_i + d^\pm_{i2}\sigma'_{i2} = \varphi^\pm_{i2}. \tag{4.10}$$

System (4.9), (4.10) of the elastic layer in the first approximation can be presented in the form of six first-order differential equations with respect to the basic functions u_0, u_1, v_0, t_{11}, m_{11}, and t_{12}

$$\begin{gathered}
t'_{11} = -[(\sigma^+_{12} - \sigma^-_{12})/2 + f^0_1]/\eta, \qquad u'_0 = (\alpha_2 t_{11} - \alpha_1\gamma_1 t_{22})/(\eta\alpha_1\alpha_2(1-\gamma_1\gamma_2)),\\
m'_{11} = [3t_{12} - 3(\sigma^+_{12} + \sigma^-_{12})/2 - f^1_1]/\eta, \quad u'_1 = (\alpha_2 m_{11} - \alpha_1\gamma_1 m_{22})/(\eta\alpha_1\alpha_2(1-\gamma_1\gamma_2)),\\
t'_{12} = -[(\sigma^+_{22} - \sigma^-_{22})/2 + f^0_2]/\eta, \qquad v'_0 = (t_{11}/m - u_1 - u_3)/\eta
\end{gathered} \tag{4.11}$$

and eight equations, which contain, in addition to the basic quantities, the additional quantities u_2, u_3, v_1, v_2, m_{12}, r_{12}, t_{22}, and m_{22}, and specified functions entering into the right sides of conditions (3.13):

$$\begin{gathered}
v_1 = (\alpha_1 t_{22} - \gamma_2\alpha_2 t_{11})/(\alpha_1\alpha_2(1-\gamma_1\gamma_2)), \qquad m_{12} = 3mu_2,\\
v_2 = (\alpha_1 m_{22} - \gamma_2\alpha_2 m_{11})/(3\alpha_1\alpha_2(1-\gamma_1\gamma_2)), \qquad r_{12} = 5mu_3,\\
c^+_{12}(u_0 + u_1 + u_2 + u_3) + d^+_{12}(t_{12} + m_{12} + r_{12}) = \varphi^+_{12}(\xi),\\
c^-_{12}(u_0 - u_1 + u_2 - u_3) + d^-_{12}(t_{12} - m_{12} + r_{12}) = \varphi^-_{12}(\xi),\\
c^+_{22}(v_0 + v_1 + v_2) + d^+_{22}(t_{22} + m_{22}) = \varphi^+_{22}(\xi),\\
c^-_{22}(v_0 - v_1 + v_2) + d^-_{22}(t_{22} - m_{22}) = \varphi^-_{22}(\xi).
\end{gathered} \tag{4.12}$$

From Eqs. (4.12), the additional quantities can be expressed in terms of the basic and known functions $c^\pm_{i2}(\xi)$, $d^\pm_{i2}(\xi)$, and $\varphi^\pm_{i2}(\xi)$ $(i = 1, 2)$. Substituting these expressions into Eqs. (4.11), we obtain a system of sixth-order differential equations with respect to the basic quantities. The order of this system is independent of the form of the boundary conditions on the layer surfaces.

If we introduce the vector $\boldsymbol{z} = [u_0, u_1, v_0, t_{11}, m_{11}, t_{12}]^{\mathrm{t}}$, the system of equations of the layer can be written as

$$\boldsymbol{z}' = H\boldsymbol{z} + \boldsymbol{F}, \tag{4.13}$$

where H is a 6×6 quadratic matrix and $\boldsymbol{F}$ is a vector of six components.

For system (4.13), for $\xi = \xi_0$ and $\xi = \xi_1$, we impose the boundary conditions of the form

$$A\boldsymbol{x} + B\boldsymbol{y} = \boldsymbol{C}, \tag{4.14}$$

where

$$\boldsymbol{x} = \begin{Vmatrix} u_0 \\ u_1 \\ v_0 \end{Vmatrix}, \qquad \boldsymbol{y} = \begin{Vmatrix} t_{11} \\ m_{11} \\ t_{12} \end{Vmatrix}.$$

In (4.14), A and B are specified matrices of the order 3×3; $\boldsymbol{C}$ is a specified vector with three components.

The matrix H of system (4.13) depends on the form of the boundary conditions on the layer surfaces. A generic solution of this system can be written for an arbitrary form of the conditions. But if the structure consists of several layers, the order of the system increases, and it becomes next to impossible to construct an analytical solution. In this case, it is reasonable to use numerical algorithms.

5. Moment Finite Element. Below, we construct a stiffness matrix of a rectangular finite element $\Omega : \{x_1^- \leqslant x_1 \leqslant x_1^+,\ x_2^- \leqslant x_2 \leqslant x_2^+\}$. We assume that the forces $t_{ij}^{\pm}$ (or the corresponding mean displacements of the faces) are set on all four faces of the element, and the bending moments $m_{11}^{\pm}$ (or the mean angles of rotation of the faces $\theta_{11}^{\pm}$) are also set on two opposite side faces. We introduce the variables

$$\xi_1 = 2[x_1 - (x_1^+ + x_1^-)/2]/h_1, \qquad \xi_2 = 2[x_2 - (x_2^+ + x_2^-)/2]/h_2,$$

where $h_1 = x_1^+ - x_1^-$ and $h_2 = x_2^+ - x_2^-$. In this case, the rectangle Ω is transformed into a square.

The stresses and displacements inside the element Ω along the ξ_2 coordinate are approximated by the segments of the Legendre polynomial series (3.3), (3.5), and (3.9). We represent the functions σ_{ij}^k in the form of segments of the Legendre polynomial series $Q_i(\xi_1)$. If we require that each term in (3.3) is represented by identical segments of the polynomial series in terms of $Q_i(\xi_1)$ and $P_k(\xi_2)$, we have to assume that

$$\sigma'_{11} = \sum_{i=0}^{1}\sum_{k=0}^{1} \sigma_{11}^{(1,k)} Q_i P_k, \qquad \sigma'_{21} = \sum_{i=0}^{1} \sigma_{21}^{(i,0)} Q_i, \qquad \sigma'_{22} = \sum_{i=0}^{1}\sum_{k=0}^{1} \sigma_{11}^{(1,k)} Q_i P_k,$$

$$\sigma'_{21} = \sum_{i=0}^{1} \sigma_{21}^{(i,0)} Q_i, \qquad f'_1 = \sum_{k=0}^{1} f_1^{(0,k)} P_k, \qquad f'_2 = f_2^{(0,0)}, \tag{5.1}$$

where

$$\sigma_{\alpha 1}^{(i,k)} = \frac{(1+2i)(1+2k)}{4} \int_{-1}^{1}\int_{-1}^{1} \sigma_{\alpha 1} Q_i P_k \, d\xi_1 \, d\xi_2, \qquad f_{\alpha}^{(0,k)} = \frac{1+2k}{4} \int_{-1}^{1}\int_{-1}^{1} f_{\alpha} P_k \, d\xi_1 \, d\xi_2, \qquad \alpha = 1, 2.$$

Representing the coefficients of the segments of the series for displacements in (3.3) and (3.15) by segments of the Legendre polynomial series $Q_i(\xi_1)$, we require that the problem of determining the expansion coefficients in the rectangle Ω has a solution for all boundary conditions for $x_1 = \pm l$ of the form

$$c_{11}^{\pm} u_1^k + d_{11}^{\pm} \sigma_{11}^k = (\varphi_{11}^{\pm})^k, \qquad c_{21}^{\pm} u_2^0 + d_{21}^{\pm} \sigma_{21}^0 = (\varphi_{21}^{\pm})^0 \qquad (k = 0, 1) \tag{5.2}$$

and the following relation is valid:

$$\frac{\partial u''_1}{\partial x_2} \sim \sigma'_{12}, \qquad \frac{\partial u''_2}{\partial x_2} \sim \sigma'_{22}.$$

Here, the tilde denotes an identical power of the polynomials along ξ_1 and ξ_2. In accordance with these requirements, we have to set

$$u'_1 = \sum_{i=0}^{2}\sum_{k=0}^{1} u_1^{(i,k)} Q_i P_k, \qquad u''_1 = \sum_{k=0}^{3} u_1^{(0,k)} P_k,$$

$$u'_2 = \sum_{i=0}^{2} u_2^{(i,0)} Q_i, \qquad u''_2 = \sum_{k=0}^{2} u_2^{(0,k)} P_k. \tag{5.3}$$

The strains inside the element are expressed in terms of displacements (5.3) by the formulas

$$\varepsilon_{11} = \frac{\partial u_1'}{\partial x_1}, \qquad 2\varepsilon_{12} = \frac{\partial u_1''}{\partial x_2} + \frac{\partial u_2'}{\partial x_1}, \qquad \varepsilon_{22} = \frac{\partial u_2''}{\partial x_2}. \tag{5.4}$$

The stresses inside the layer are calculated by the formulas

$$\sigma_{ij} = a_{ijmn}\varepsilon_{mn}, \tag{5.5}$$

where ε_{mn} are determined by relations (5.4).

Equations (3.7), (5.4), and (5.5) and the boundary conditions

$$u_1'(x_1^{\pm}) = u_{11}^{\pm} \pm h_2\theta_{11}^{\pm}P_1/2, \qquad u_2'(x_1^{\pm}) = u_{21}^{\pm}, \qquad u_i'(x_2^{\pm}) = u_{i2}^{\pm} \quad (i = 1, 2)$$

form a closed system of algebraic equations with respect to the coefficients $u_\alpha^{(i,k)}$ and $\sigma_m^{(i,k)}$ entering into expansions (5.1) and (5.3). The generalized forces and displacements are determined by relations (4.4). Solving this system, we find the expressions for the forces $t_{ij}^{\pm}$ and moments $m_{11}^{\pm}$ via the displacements $u_{ij}^{\pm}$ and angles of rotation $\theta_{11}^{\pm}$ of the element faces. The forces $t_{ij}^{\pm}$ and moments $m_{11}^{\pm}$ are related to the solution of the system by the equalities

$$t_{i1}^{\pm} = h_2(\sigma_{i1}^{(0,0)} \pm \sigma_{i1}^{(1,0)}), \qquad t_{i2}^{\pm} = h_1\sigma_{i2}'(\pm 1) \qquad (i = 1, 2),$$

$$m_{11}^{\pm} = h_2^2(\sigma_{11}^{(0,1)} \pm \sigma_{11}^{(1,1)})/6.$$

We introduce the notation

$$u_{ij}^0 = (u_{ij}^+ + u_{ij}^-)/2, \quad u_{ij}^1 = u_{ij}^+ - u_{ij}^-, \quad \theta_{11}^0 = (\theta_{11}^+ + \theta_{11}^-)/2, \quad \theta_{11}^1 = \theta_{11}^+ - \theta_{ij}^-. \tag{5.6}$$

Note, the linear combinations of quantities (5.6)

$$u_{11}^1,\ u_{22}^1,\ u_{12}^1/h_2 + u_{21}^1/h_1,\ \theta_{11}^1,\ u_{12}^1/h_2 - \theta_{11}^0,\ u_{12}^0 - u_{11}^0,\ u_{21}^0 - u_{22}^0 \tag{5.7}$$

are equal to zero if the element is displaced as a stiff whole.

To find the coefficients of the element stiffness matrix, we have to perform the following calculations. Using (5.3) and (5.6), we express $u_\alpha^{(i,k)}$ via $u_{11}^{(0,0)}$, $u_{22}^{(0,0)}$, $u_{11}^{(0,1)}$, and quantities (5.6). From Eqs. (3.7), (5.4), and (5.5), we express the quantities $u_{11}^{(0,0)}$, $u_{22}^{(0,0)}$, and $u_{11}^{(0,1)}$ and strains via quantities (5.7). If relations (5.5) in the case of a two-dimensional problem are written in the form

$$\sigma_{11} = a\varepsilon_{11} + b\varepsilon_{22}, \qquad \sigma_{22} = b\varepsilon_{11} + a\varepsilon_{22}, \qquad \sigma_{12} = 2\mu\varepsilon_{12},$$

we obtain after the calculations

$$t_{11}^0 = a\,\frac{h_2}{h_1}\,u_{11}^1 + bu_{22}^1, \quad t_{22}^0 = h_1\Big(\frac{a}{h_2}\,u_{22}^1 + \frac{b}{h_1}\,u_{11}^1\Big), \quad t_{21}^0 = \mu h_2\Big(\frac{1}{h_1}\,u_{21}^1 + \frac{1}{h_2}\,u_{12}^1\Big),$$

$$t_{12}^0 = \mu h_1\Big\{\frac{1}{h_1}\,u_{21}^1 + \frac{1}{h_2}\,u_{12}^1 + \frac{5}{ah_2/h_1 + 5\mu h_1/h_2}\Big[a\,\frac{h_2}{h_1}\Big(\frac{1}{h_2}\,u_{12}^1 - \theta_{11}^0\Big) - \frac{1}{6}\,h_1 f_1^{(0,1)}\Big]\Big\},$$

$$t_{11}^1 = \frac{12a}{\mu h_1/h_2 + ah_2/h_1}\Big[\mu(u_{11}^0 - u_{12}^0) - \frac{1}{12}\,h_2^2 f_1^{(0,0)}\Big], \tag{5.8}$$

$$t_{12}^1 = \frac{12\mu}{\mu h_1/h_2 + ah_2/h_1}\Big[a(u_{12}^0 - u_{11}^0) - \frac{1}{12}\,h_1^2 f_1^{(0,0)}\Big],$$

$$t_{21}^1 = \frac{\mu}{\mu h_2/h_1 + ah_1/h_2}\Big[12a(u_{21}^0 - u_{22}^0) - \frac{bh_2^2}{h_1}\,\theta_{11}^1 - h_2^2 f_2^{(0,0)}\Big],$$

$$t_{22}^1 = \frac{\mu}{\mu h_2/h_1 + ah_1/h_2}\Big[12a(u_{22}^0 - u_{21}^0) + \frac{bh_2^2}{h_1}\,\theta_{11}^1 - \frac{ah_1^2}{\mu}\,f_2^{(0,0)}\Big],$$

$$m_{11}^1 = \frac{ah_2^2}{ah_2/h_1 + 5\mu h_1/h_2}\Big[5\mu\theta_{11}^0 - \frac{1}{6}\,h_2 f_1^{(0,1)}\Big],$$

$$m_{11}^0 = \frac{h_2^2}{6(\mu h_2/h_1 + ah_1/h_2)}\Big\{\frac{6b\mu}{h_1}\,(u_{22}^0 - u_{21}^0) + \frac{1}{2}\Big[a\mu\,\frac{h_2^2}{h_1} + a^2 - b^2\Big]\theta_{11}^1 - \frac{1}{2}\,bh_1 f_2^{(0,0)}\Big\}.$$

Expressions (5.8) can be represented as

$$t_{ij}^0 = \langle t_{ij}^0 \rangle + c_{ij}^0, \quad t_{ij}^1 = \langle t_{ij}^1 \rangle + c_{ij}^1, \quad m_{11}^0 = \langle m_{11}^0 \rangle + d_{11}^0, \quad m_{11}^1 = \langle m_{22}^1 \rangle + d_{11}^1, \tag{5.9}$$

where $\langle t_{ij}^0 \rangle$, $\langle t_{ij}^1 \rangle$, $\langle m_{11}^0 \rangle$, and $\langle m_{11}^1 \rangle$ are linear combinations of quantities (5.7). Hence, $\langle t_{ij}^0 \rangle$, $\langle t_{ij}^1 \rangle$, $\langle m_{11}^0 \rangle$, and $\langle m_{11}^1 \rangle$ vanish if the element is displaced as a stiff whole, and the quantities c_{ij}^0, c_{ij}^1, d_{11}^0, and m_{11}^1 are linear combinations of f_1' and f_2'.

Relations (5.9) determine the stiffness matrix of the moment rectangular element.

Since

$$\int\limits_\Omega a_{ijmn}\varepsilon_{ij}\varepsilon_{mn}\, d\Omega = \int\limits_\Omega \Big[f_i'u_i' + \frac{\partial}{\partial x_1}\,(\sigma_{i1}'u_i') + \frac{\partial}{\partial x_2}\,(\sigma_{i2}'u_i'')\Big]\, d\Omega = t_{ij}^+u_{ij}^+ - t_{ij}^-u_{ij}^- = \langle t_{ij}^1 \rangle u_{ij}^0 + \langle t_{ij}^0 \rangle u_{ij}^1, \tag{5.10}$$

then $\langle t_{ij}^1 \rangle u_{ij}^0 + \langle t_{ij}^0 \rangle u_{ij}^1 \geqslant 0$. The equality in (5.10) is only possible if the element is displaced as a stiff whole. This ensures the uniqueness of the solution for all boundary conditions of the form (5.2).

The procedure of calculating the global stiffness matrix for a domain constructed of rectangles and the iteration algorithm for determining the stress-strain state for such regions can be found in [8].

In the above-described approach to constructing the elastic layer equations, the representations of stresses and displacements in the form of segments of the Legendre polynomials were actually used to approximate the derivatives of stresses and displacements entering into the equations. The stresses inside the layer are calculated by formulas (3.14). This allows one to generalize the approach described above to the case of constructing equations of the layer whose behavior is described by physically nonlinear governing relations.

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REFERENCES

1. S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, McGraw-Hill, New York (1959).
2. A. I. Soler, “Higher-order theories for structural analysis using Legendre polynomial expansion,” *Trans. ASME, Ser. E, J. Appl. Mech.*, **36**, No. 4 (1969).
3. G. V. Ivanov, “Solution of the plane mixed problem of the theory of elasticity in the form of a series in Legendre polynomials,” *J. Appl. Mech. Tech. Phys.*, **17**, No. 6, 856–866 (1976).
4. L. A. Dergileva, “Method for solving a plane contact problem for an elastic layer,” in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 25, Inst. Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR (1976), pp. 24–32.
5. Yu. M. Volchkov and L. A. Dergileva, “Solution of problems of an elastic layer on the basis of approximate equations and comparison with the solutions of the elasticity theory,” in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 28, Inst. Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR (1977), pp. 43–54.
6. Yu. M. Volchkov and L. A. Dergileva, “Edge effects in the stress state of a thin elastic interlayer,” *J. Appl. Mech. Tech. Phys.*, **40**, No. 2, 354–359 (1999).
7. A. E. Alekseev, “Bending of a three-layer orthotropic beam,” *J. Appl. Mech. Tech. Phys.*, **36**, No. 3, 458–465 (1995).
8. Yu. M. Volchkov, L. A. Dergileva, and G. V. Ivanov, “Numerical simulation of stress states in plane problems of elasticity by the method of layers,” *Prikl. Mekh. Tekh. Fiz.,*, **35**, No. 6, 354–359 (1994).
9. A. E. Alekseev, “Equations of deformation of an elastic layer of variable thickness,” in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 81, Inst. Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR (1987), pp. 3–13.